Intro Video: Section 3.9
Related Rates

Math F251X: Calculus I

Idea: • We have some function $f(t)$ that depends on t

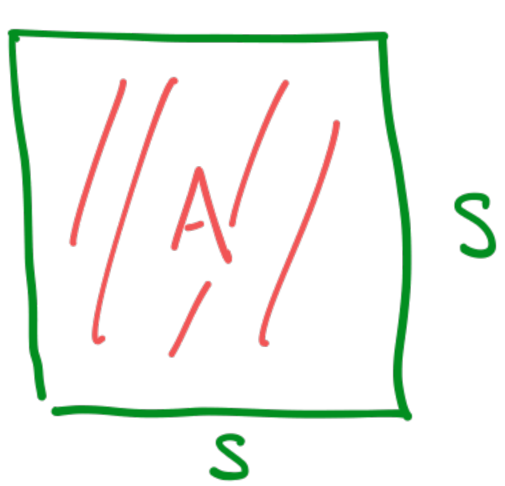
• We have some function $H(t)$ that also depends on t.

• f and H are related by some function.

• If we know information about $\frac{df}{dt}$, we can use it to say something about $\frac{dH}{dt}$.

Example: If the side length s of a square is increasing at a rate of 3 cm/s, how fast is the area A increasing when $s = 12$?

Know $\frac{ds}{dt} = 3$ cm/s   WANT $\frac{dA}{dt}$ when $s = 12$.



Know $A = s^2 \Longrightarrow \frac{d}{dt}(A) = \frac{d}{dt}(s^2)$

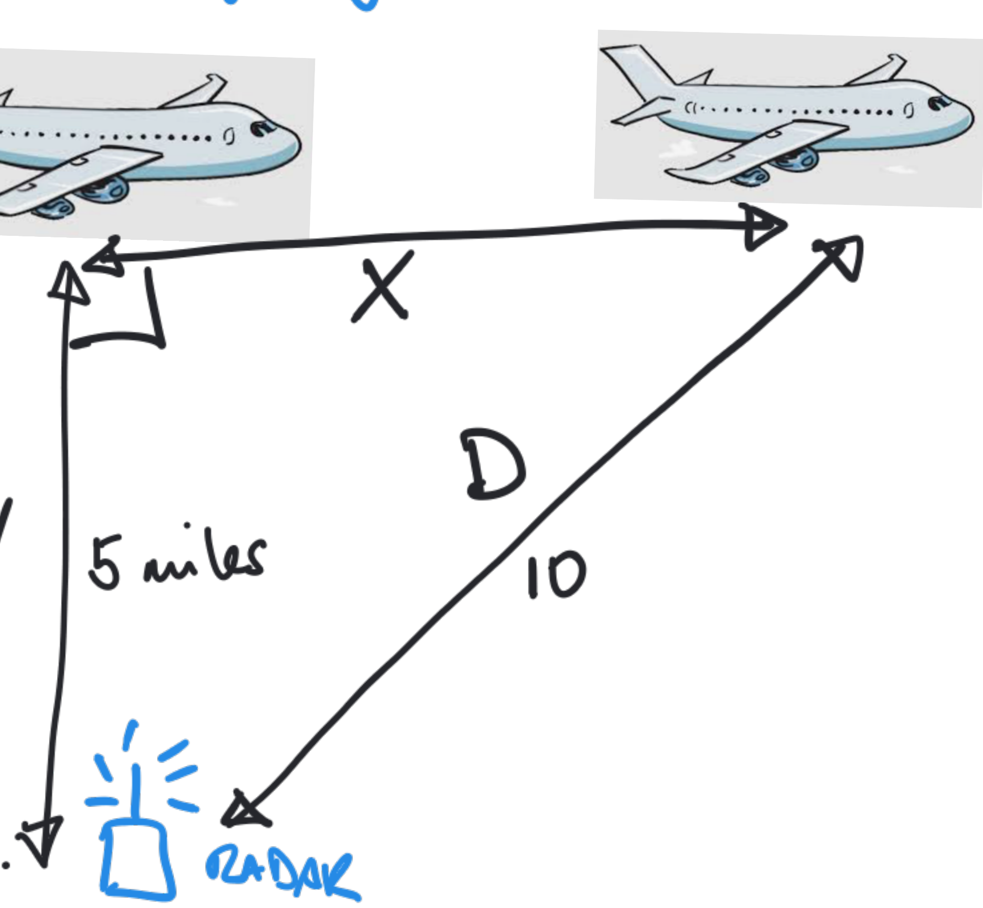
$\Longrightarrow \frac{dA}{dt} = 2s\frac{ds}{dt}$. So $\frac{dA}{dt} = 2(12)(3) = 72\ \frac{cm^2}{s}$

cm   cm/s

STRATEGY:

1. Draw a picture and label useful stuff.
2. Identify what you KNOW
3. Identify what you WANT
4. Relate (with an equation) what you KNOW and WANT
5. Implicitly differentiate with respect to time
6. Substitute in what you KNOW
7. Solve for what you WANT.

Example: An airplane is flying at an altitude of 5 miles, and passes directly over a radar antenna. When the plane is 10 miles away from the radar antenna, the radar detects that the distance is changing at a rate of 240 miles per hour. How fast is the plane going?



KNOW: $\frac{dD}{dt} = 240 \text{ mi/h}$ when $D = 10$.

WANT: $\frac{dX}{dt}$.

RELATE: $5^2 + X^2 = D^2$.

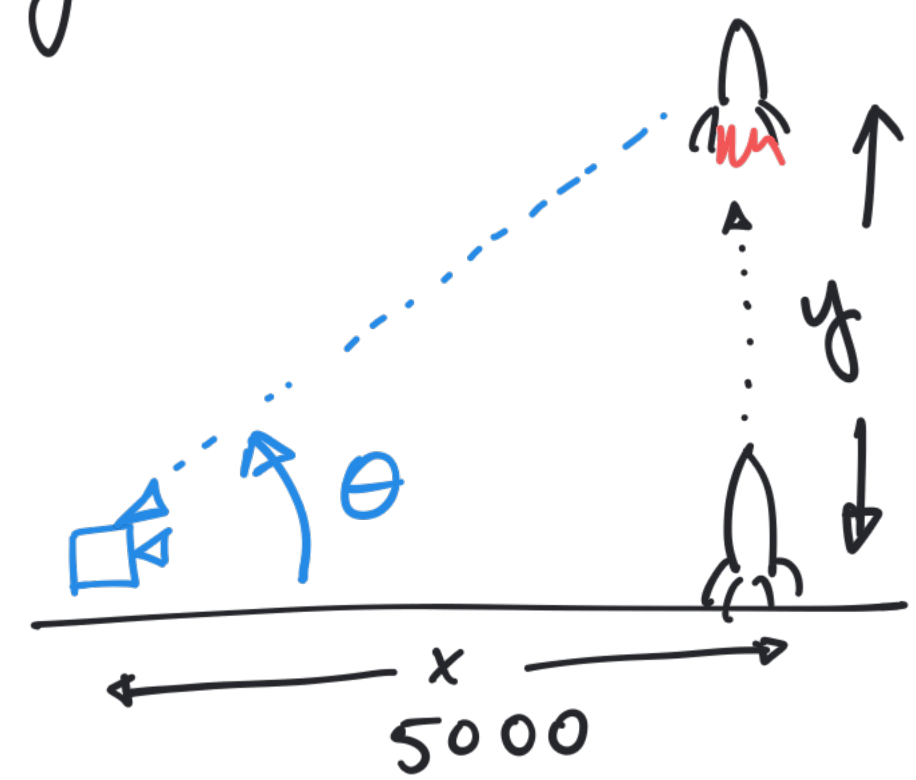
So $2X \frac{dX}{dt} = 2D \frac{dD}{dt}$. When $D = 10$,

$5^2 + X^2 = 100 \Rightarrow X = \sqrt{75}$.

So $2\sqrt{75}\frac{dX}{dt} = 2(10)(240)$

$\Rightarrow \frac{dX}{dt} = \frac{2400}{\sqrt{75}} \approx 277 \text{ mph.}$

Example: A rocket is launched. A camera is 5000 ft from the launchpad. When the rocket is 1000 ft above the launchpad, its velocity is 600 ft/s. How fast does the camera angle need to change so that it can stay focused on the rocket?



KNOW

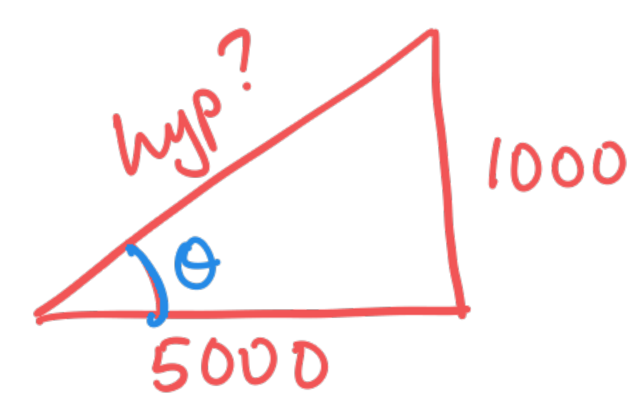
$x = 5000$

When $y = 1000$, $\frac{dy}{dt} = \boxed{600}$ ft/s

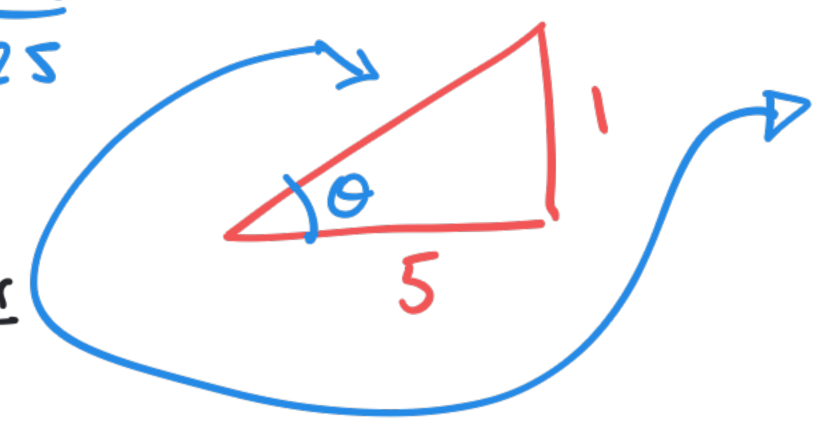
Want $\frac{d\theta}{dt}$

Relate: $\frac{y}{5000} = \tan\theta \Rightarrow y = 5000\tan\theta$

So $\frac{dy}{dt} = 5000(\sec(\theta))^2 \frac{d\theta}{dt}$



When $y = 1000$, the hypotenuse is $1000\sqrt{26}$, so $(\sec(\theta))^2 = \left(\frac{1000\sqrt{26}}{5000}\right)^2 = \frac{26}{25}$



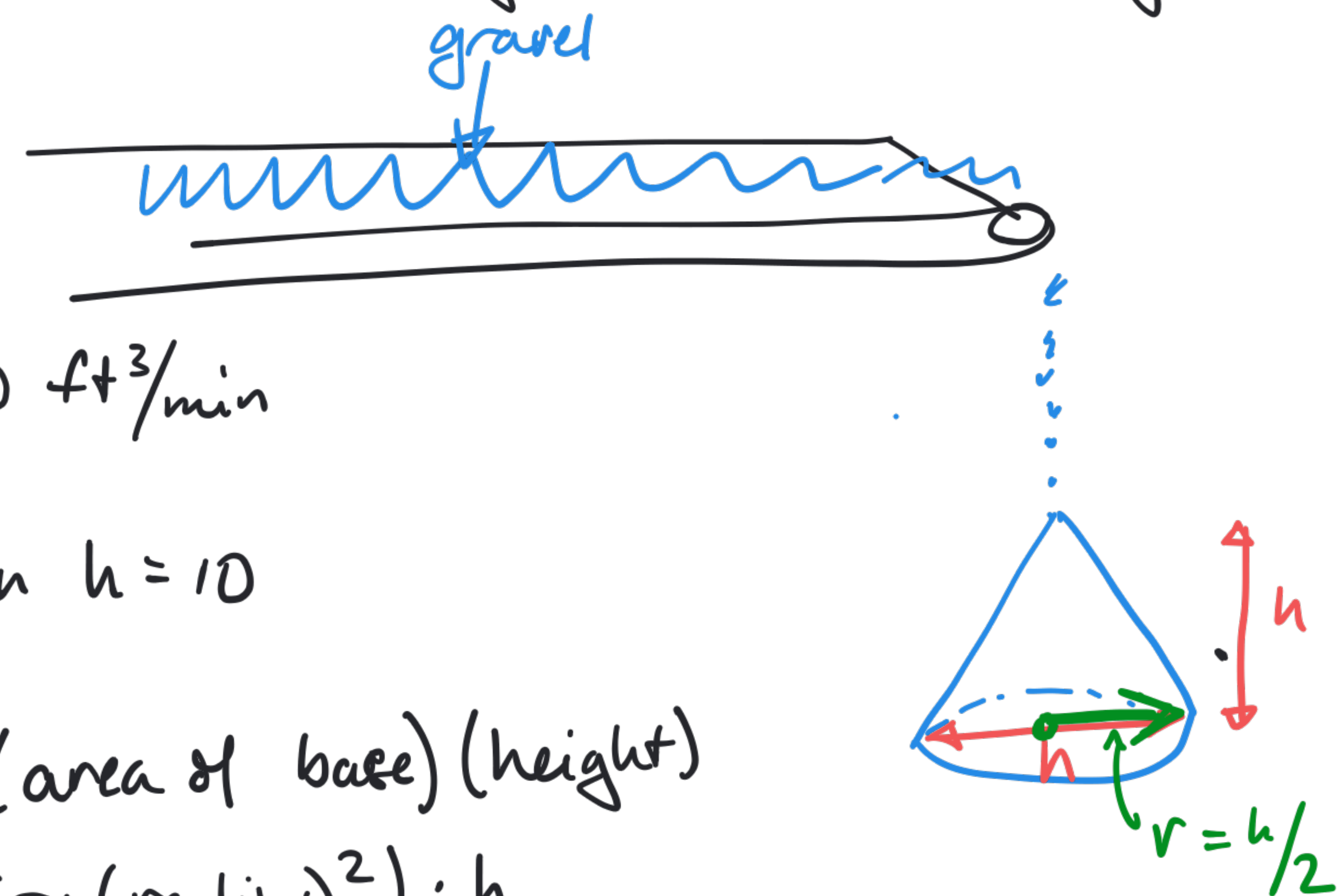
$= \sqrt{1^2 + 5^2}$

$= \sqrt{26}$

So $\boxed{600} = 5000\left(\frac{26}{25}\right)\frac{d\theta}{dt} \Rightarrow$

$$\frac{d\theta}{dt} = \frac{600}{5000\cdot\frac{26}{25}} = \frac{6\cdot\cancel{25}}{\underset{2}{\cancel{50}}\cdot 26} = \frac{6}{52} = \frac{3}{26}\ \frac{\text{radians}}{\text{s}}$$

Example: Gravel pours off a conveyor belt onto a cone-shaped pile. Observationally, the height and base-diameter of the cone are always equal. If the gravel is being dumped out at a rate of 30 ft³/min, how fast is the height of the pile increasing when the pile is 10' high?



Know $\frac{dV}{dt} = 30 \text{ ft}^3/\text{min}$

WANT $\frac{dh}{dt}$ when $h = 10$

Relate $V = \frac{1}{3}(\text{area of base})(\text{height})$

$= \frac{1}{3}(\pi(\text{radius})^2) \cdot h$

$$V = \frac{1}{3}\left(\pi\left(\frac{h}{2}\right)^2\right)h = \frac{\pi}{3} \cdot \frac{h^3}{4} = \frac{\pi}{12} h^3$$

$\frac{dV}{dt} = \frac{\pi}{12}(3h^2)\frac{dh}{dt} \Rightarrow$ when $h = 10$ and $\frac{dV}{dt} = 30$,

$$30 = \frac{\pi}{12}(3(10)^2)\frac{dh}{dt} \Rightarrow \frac{dh}{dt} = \frac{3\cancel{0} \cdot \cancel{12}^{\,2}}{\underset{5}{\cancel{300}}\,\pi} = \frac{6}{5\pi} \text{ ft/min}$$